Brownian motion and Stochastic Calculus
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Assignment 2

Exercise 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X, Y, and Z be random variables and suppose that Z is $\sigma(X, Y)$-measurable. Use the monotone class theorem to show that there exists a measurable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $Z = f(X, Y)$.

Exercise 2

Fix two measurable processes X and Y on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1) Assume that X and Y are both right-continuous or both left-continuous. Show that they are $\mathbb{P}$-modifications of each other if and only if they are $\mathbb{P}$-indistinguishable.

2) Show that the previous result is not true in general.

Exercise 3

Let X be a process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ satisfies the usual conditions. We want to show

$$X \ \mathbb{F}\text{-optional} \Longrightarrow X \ \mathbb{F}\text{–progressively measurable} \Longrightarrow X \ \mathbb{F}\text{-adapted and measurable}.$$

1) Show that every $\mathbb{F}$-progressively measurable process is $\mathbb{F}$-adapted and measurable.

2) Assume that X is $\mathbb{F}$-adapted and that every path of X is right-continuous (resp. left-continuous). Show that X is $\mathbb{F}$–progressively measurable.

3) Show that $\mathcal{O}(\mathbb{F})$ is generated by all bounded, càdlàg, $\mathbb{F}$-adapted and measurable processes.

4) Use the monotone class theorem to show that every $\mathbb{F}$-optional process is $\mathbb{F}$–progressively measurable.

Exercise 4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and B a $\mathbb{P}$–Brownian motion on $[0, 1]$. Let $k \in \mathbb{N}^\star$, and $0 = s_1 < t_1 < s_2 < t_2 < \cdots < t_k < s_{k+1} = 1$. Find the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditional on $(B_{s_1}, \ldots, B_{s_{k+1}})$.

Let $D := \big\{a2^{-m} : m \in \mathbb{N},\ a \in \{0, 1, \ldots, 2^m\}\big\}$. Let Z_1, Z_2,... be an infinite sequence of i.i.d. standard normal random variables. Construct in terms of the Z_j a stochastic process $(W_t)_{t \in D}$ such that the law of W is equal to the law of $(B_t)_{t \in D}$.